

# The Planck Blackbody Spectrum Follows from the Structure of Relativistic Spacetime

Timothy H. Boyer

*Department of Physics, City College of the City  
University of New York, New York, New York 10031*

## Abstract

Here we show that within classical physics, the Planck blackbody spectrum can be derived directly from the structure of relativistic spacetime. In noninertial frames, thermal radiation at positive temperature is connected directly to zero-point radiation whose spectrum follows from the geodesic structure of the spacetime. The connection between zero-point radiation and thermal radiation at positive temperature is through a time-dilating conformal transformation in the non-inertial frame. Transferring the spectrum back to Minkowski spacetime, the Planck spectrum is obtained.

In textbooks of physics, the Planck blackbody radiation spectrum is said to require quantum theory for its derivation. Actually, the Planck blackbody spectrum can be derived directly from the structure of relativistic spacetime within classical physics. In noninertial frames, thermal radiation at positive temperature is connected directly to zero-point radiation whose spectrum follows from the geodesic structure of the spacetime.

The derivation of the Planck spectrum for relativistic radiation depends upon the following essential ideas. Zero-point radiation corresponds to the spectrum of random classical radiation which is as featureless as possible; its correlation function depends upon only the geodesic separation between the points at which the correlation function is evaluated. A Minkowski frame in flat spacetime is such a specialized, featureless system that Minkowski spacetime gives no structure to the zero-point spectrum; in a Minkowski frame, zero-point radiation is carried into itself by a time-dilating symmetry transformation. However, in a static noninertial frame, each spatial point undergoes a proper acceleration relative to the momentarily comoving reference frame, and this acceleration gives a structure to the zero-point correlation function; under a time-dilating symmetry transformation, zero-point radiation is carried into thermal radiation at positive temperature. The thermal radiation spectrum in the noninertial frame can then be carried back into an inertial frame. The result is the Planck spectrum for blackbody radiation in a Minkowski frame derived from fundamental ideas of spacetime structure. In this article, we carry out the explicit calculations for relativistic scalar radiation in a spacetime of four dimensions.

In a Minkowski frame in flat spacetime, the familiar spacetime coordinates  $ct, x, y, z$  are geodesic coordinates with interval  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - dx^2 - dy^2 - dz^2$ . Thus the geodesic separation between two spacetime points (primed and unprimed) is simply  $s^2 = g_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu) = c^2(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2$ . The relativistic scalar field  $\phi$  with Lagrangian density  $\mathcal{L} = [1/(8\pi)]g^{\mu\nu}(\partial\phi/\partial x^\mu)(\partial\phi/\partial x^\nu)$  has scaling dimension one in four spacetime dimensions. Accordingly, the correlation function for the zero-point radiation of the relativistic scalar field must take the form

$$\langle\phi_0(ct, x, y, z)\phi_0(ct', x', y', z')\rangle = \frac{-(\hbar c/\pi)}{c^2(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2} \quad (1)$$

where the constant in the numerator has been chosen so as to connect with the familiar spectrum of classical zero-point radiation corresponding to an energy  $\mathcal{E}_0(\omega) = (1/2)\hbar\omega$

per normal mode.[1][2] We notice that the correlation function depends simply upon the geodesic separation between the spacetime points, and there is no characteristic time or length in this correlation function; the one constant  $(\hbar c/\pi)$  which appears gives an overall scale. The appearance of the constant  $\hbar$  has nothing to do with quantum theory and is merely a constant taking the numerical value of Planck's constant so as to give agreement with experimental measurements of Casimir forces.[1]

The correlation function for zero-point radiation in Eq. (1) is invariant under a time-dilating symmetry transformation. The Minkowski spacetime metric undergoes a scale transformation under the time-dilating transformation  $t \rightarrow \bar{t} = \sigma t$  where  $\sigma$  is a positive numerical constant, provided that the space coordinates are also dilated giving the full transformation

$$t \rightarrow \bar{t} = \sigma t, \quad x \rightarrow \bar{x} = \sigma x, \quad y \rightarrow \bar{y} = \sigma y, \quad z \rightarrow \bar{z} = \sigma z, \quad (2)$$

so that the spacetime interval is rescaled as

$$d\bar{s}^2 = \sigma^2 ds^2 = \sigma^2 (c^2 dt^2 - dx^2 - dy^2 - dz^2). \quad (3)$$

If we interpret this time-dilating symmetry transformation in the active sense as a transformation of the field  $\phi$  in the old coordinates, then we can write the transformation as  $\phi \rightarrow \bar{\phi}$  where  $\bar{\phi}(ct, x, y, z) = \sigma \phi(c\sigma t, \sigma x, \sigma y, \sigma z)$ . The correlation function for zero-point radiation  $\phi_0$  is invariant under this transformation[3] since

$$\begin{aligned} \langle \bar{\phi}_0(ct, x, y, z) \bar{\phi}_0(ct', x', y', z') \rangle &= \langle \sigma \phi_0(c\sigma t, \sigma x, \sigma y, \sigma z) \sigma \phi_0(c\sigma t', \sigma x', \sigma y', \sigma z') \rangle \\ &= \frac{-\sigma^2(\hbar c/\pi)}{c^2(\sigma t - \sigma t')^2 - (\sigma x - \sigma x')^2 - (\sigma y - \sigma y')^2 - (\sigma z - \sigma z')^2} \\ &= \frac{-(\hbar c/\pi)}{c^2(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2}. \end{aligned} \quad (4)$$

In contrast to zero-point radiation, equilibrium thermal radiation at positive temperature in a Minkowski frame will involve a parameter corresponding to the temperature  $T$  in the unique inertial frame in which the radiation spectrum is isotropic. Now within classical physics, the zero-point radiation ( $\mathcal{E}_0(\omega) = (1/2)\hbar\omega$  per normal mode) is regarded as radiation which is always present; thermal radiation ( $\mathcal{E}_T(\omega, T)$  per normal mode), on the other hand, involves additional radiation energy per normal mode above the zero-point energy per normal mode,  $\mathcal{E}_T > \mathcal{E}_0$ . Since thermal radiation involves a finite total radiation energy density above the zero-point radiation for a system of an infinitely many normal

modes, the smooth thermal spectrum must involve a transition frequency  $\omega_{tr}$  where the zero-point radiation energy per normal mode (which vanishes at low frequency and increases steadily) is equal to the thermal energy per normal mode of the lowest frequency modes,  $\mathcal{E}_T(0, T) = k_B T = \mathcal{E}_0(\omega_{tr}) = (1/2)\hbar\omega_{tr}$ ; therefore  $\omega_{tr} = 2k_B T/\hbar$ . This transition frequency corresponds roughly to the frequency in the spectrum where the functional form changes from the low-frequency to the high-frequency form. The correlation function for thermal radiation will contain a transition time  $t_{tr} = 2\pi/\omega_{tr}$  which reflects this transition frequency. Under the time-dilating symmetry which rescales the radiation fields as  $\bar{\phi}(ct, x, y, z) = \sigma\phi(c\sigma t, \sigma x, \sigma y, \sigma z)$ , the thermal spectrum will be transformed from positive temperature  $T$  over to positive temperature  $\bar{T} = \sigma T$ , and the transition frequency  $\omega_{tr}$  and transition time  $t_{tr}$  will be transformed to  $\bar{\omega}_{tr} = \sigma\omega_{tr}$  and  $\bar{t}_{tr} = t_{tr}/\sigma$ . We notice that for any positive dilation factor  $\sigma$ , the new temperature  $\bar{T}$ , the new transition frequency  $\bar{\omega}_{tr}$ , and the new transition time  $\bar{t}_{tr}$  are always positive. Thus under a time-dilating symmetry transformation, the set of thermal correlation functions and the set of thermal spectral functions for finite temperature in a Minkowski frame are carried into themselves, but the zero-point correlation function and zero-point radiation spectrum are not included in these sets. The zero-point correlation function and zero-point spectrum are the limiting functions obtained from the thermal radiation functions at positive temperature as  $\sigma \rightarrow 0$ . Crucially, we cannot go back from this  $\sigma \rightarrow 0$ -limit to obtain thermal radiation at positive temperature from zero-point radiation at zero temperature in a Minkowski frame. The Minkowski frame is such a specialized system that its coordinates are geodesic coordinates, and thermal radiation cannot be obtained from zero-point radiation by a time-dilating symmetry transformation.

The situation is quite different for a static coordinate frame in a gravitational field or for a static noninertial coordinate frame in flat spacetime. In a static coordinate frame, none of the metric components  $g_{\mu\nu}$  are functions of the time coordinate. Although relativistic radiation follows the light-like geodesics in a relativistic spacetime, the points with fixed spatial coordinates in a noninertial frame do not follow geodesics. In a static coordinate frame with time coordinate  $\eta$ , we have  $ds^2 = g_{00}(x^i)d\eta^2 + g_{ij}(dx^j)(dx^i)$  with  $\partial_0 g_{\mu\nu} = \partial_\eta g_{\mu\nu} = 0$  where the indices  $i$  and  $j$  label the spatial coordinates. Thus in a static noninertial frame, the geodesic equation  $d^2x^\mu/d\tau^2 + \Gamma_{\rho\sigma}^\mu(dx^\rho/d\tau)(dx^\sigma/d\tau) = 0$  for a particle instantaneously at

rest  $(dx^i/d\eta) = 0$  at spatial coordinates  $x^i$  becomes[4]

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma_{00}^\mu \left( \frac{d\eta}{d\tau} \right)^2 = \frac{d^2x^\mu}{d\tau^2} + \left( -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \right) \frac{c^2}{g_{00}}. \quad (5)$$

Thus a point with fixed spatial coordinates  $x^i$  in the static coordinate frame has a constant proper acceleration given by  $-(1/2)g^{\mu\lambda}(\partial_\lambda g_{00})c^2/g_{00}$ , which is the negative of the geodesic acceleration in Eq. (5).

A Rindler frame in flat spacetime is an example of a simple static noninertial frame. In a Rindler frame where

$$ds^2 = \xi^2 d\eta^2 - d\xi^2 - dy^2 - dz^2, \quad (6)$$

the proper acceleration  $a(\xi)$  of a point with coordinates  $x^1 = \xi$ ,  $x^2 = y$ ,  $x^3 = z$  is in the direction of the coordinate  $x^1 = \xi$ , and is given by

$$a(\xi) = -\frac{1}{2}g^{11}(\partial_1 g_{00})c^2/g_{00} = -\frac{1}{2}(-1)(2\xi)\frac{c^2}{\xi^2} = \frac{c^2}{\xi} \quad (7)$$

This constant proper acceleration of the spatial point relative to the geodesic coordinates of a fixed Minkowski frame gives the relation between the coordinates of the noninertial frame relative to geodesic coordinates as[5]

$$ct = \xi \sinh \eta, \quad x = \xi \cosh \eta, \quad (8)$$

while the  $y$  and  $z$  coordinates are unchanged between the Minkowski and Rindler coordinate frames.

Since we are dealing here with a scalar radiation field  $\phi(ct, x, y, z)$ , the field  $\varphi(\eta, \xi, y, z)$  in the Rindler frame takes the same value at the corresponding point of the Rindler frame,

$$\phi(ct, x, y, z) = \phi(\xi \sinh \eta, \xi \cosh \eta, y, z) = \varphi(\eta, \xi, y, z). \quad (9)$$

Therefore the correlation function for the zero-point field  $\varphi_0$  in the Rindler frame follows from Eq. (1) as

$$\begin{aligned} & \langle \varphi_0(\eta, \xi, y, z) \varphi_0(\eta', \xi', y', z') \rangle \\ &= \langle \phi_0(\xi \sinh \eta, \xi \cosh \eta, y, z) \phi_0(\xi' \sinh \eta', \xi' \cosh \eta', y', z') \rangle \\ &= \frac{-(\hbar c/\pi)}{(\xi \sinh \eta - \xi' \sinh \eta')^2 - (\xi \cosh \eta - \xi' \cosh \eta')^2 - (y - y')^2 - (z - z')^2} \\ &= \frac{-(\hbar c/\pi)}{2\xi\xi' \cosh(\eta - \eta') - \xi^2 - \xi'^2 - (y - y')^2 - (z - z')^2}. \end{aligned} \quad (10)$$

The zero-point correlation function (10) still depends upon the geodesic separation between the spacetime points at which it is evaluated, but now the coordinates used in the evaluation are those of the noninertial Rindler frame. The Rindler zero-point correlation function in Eq. (10) at a single time  $\eta = \eta'$  takes exactly the same form  $|\mathbf{r} - \mathbf{r}'|^{-2}$  (involving the spatial separation) as the Minkowski zero-point correlation function in Eq. (1) at the single time  $t = t'$ , because a single time in the Rindler frame is also a single time in the Minkowski frame which is instantaneously at rest with respect to the coordinates of the Rindler frame, and the Rindler spatial coordinates can be used as geodesic coordinates in the associated Minkowski frame.

It is the concept of time which is so vastly different between the Rindler frame (or any noninertial frame) and any Minkowski frame. The correlation function for the zero-point field  $\varphi_0$  at a single spatial point  $\xi, y, z$  in the Rindler frame but at two different times  $\eta, \eta'$  follows from Eq. (10) and gives

$$\begin{aligned} \langle \varphi_0(\eta, \xi, y, z) \varphi_0(\eta', \xi, y, z) \rangle &= \frac{-(\hbar c/\pi)}{2\xi^2 \cosh(\eta - \eta') - 2\xi^2} \\ &= \frac{-(\hbar c/\pi)}{4\xi^2 \sinh^2[(\eta - \eta')/2]} \\ &= \frac{-(\hbar c/\pi)}{4\xi^2 \sinh^2[c(\tau - \tau')/(2\xi)]} \end{aligned} \quad (11)$$

where in the last line of Eq. (11) we have introduced the local proper time  $\tau = \xi\eta/c$  at the spatial coordinates  $\xi, y, z$ . We notice that for small time intervals,  $\eta - \eta' \ll 1$ , this correlation function takes the form  $\langle \varphi_0(\eta, \xi, y, z) \varphi_0(\eta', \xi, y, z) \rangle = -(\hbar c/\pi)/[c^2(\tau - \tau')^2]$  where  $\tau - \tau' = \xi(\eta - \eta')/c$  is the proper-time difference between the points. This correlation function expression involving  $(\tau - \tau')^{-2}$  agrees with the corresponding expression in Eq. (1) involving  $(t - t')^{-2}$  in the Minkowski frame. However, at large time intervals,  $\eta - \eta' \gg 1$ , the functional form in (11) changes completely and involves exponentials of the time interval. Thus in this noninertial Rindler frame, the zero-point correlation function (10) now has acquired a feature relative to the time coordinate  $\eta$  in the Rindler frame. The time interval  $\eta_{tr} = \eta - \eta' = 1$  can be taken as a transition time interval for the correlation function when evaluated in the coordinates of the Rindler frame. Associated with this transition time  $\eta_{tr} = 1$ , there will be a transition frequency  $\kappa_{tr} = 1/\eta_{tr} = 1$ . Now when we apply a time-dilating symmetry transformation to the zero-point fields, the correlation function (11) for the zero-point fields is no longer an invariant but rather is carried into a new correlation

function associated with a new transition time and a new transition frequency. This new correlation function for  $\sigma > 1$  corresponds to a situation of thermal radiation equilibrium at positive empirical temperature in the noninertial frame.

In a Minkowski frame, a time-dilating symmetry transformation  $t \rightarrow \bar{t} = \sigma t$  corresponds to a uniform scale transformation in space and time as given above in Eq. (2) with the corresponding scale change of the metric in Eq. (3). However, in a noninertial frame, a time-dilating symmetry transformation  $\eta \rightarrow \bar{\eta} = \sigma \eta$  for the radiation fields involves complicated transformations of the spatial coordinates; a time-dilating symmetry transformation in a noninertial frame is a time-dilating *conformal* transformation, where for Rindler coordinates, the metric is transformed as

$$ds^2 \rightarrow d\bar{s}^2 = \sigma^2 [f(\sigma, \xi)]^2 (\xi^2 d\eta^2 - d\xi^2 - dy^2 - dz^2). \quad (12)$$

In a Rindler frame in *two* spacetime dimensions, the time-dilating conformal transformation has been given explicitly.[6] Here, where we are working in *four* spacetime dimensions, we merely note that a time-dilating conformal transformation will transform the radiation field  $\varphi$  by the local scaling factor  $\mathfrak{s} = \sigma f(\sigma, \xi)$  in Eq. (12) while the spatial coordinate  $\xi$  is transformed as  $\xi \rightarrow \bar{\xi} = f(\sigma, \xi)\xi$  and the time coordinate as  $\eta \rightarrow \bar{\eta} = \sigma \eta$  so that the correlation function transforms as

$$\begin{aligned} \langle \bar{\varphi}_\sigma(\eta, \xi, y, z) \bar{\varphi}_\sigma(\eta', \xi, y, z) \rangle &= \frac{-[\sigma f(\sigma, \xi)]^2 (\hbar c / \pi)}{4[f(\sigma, \xi)]^2 \xi^2 \sinh^2[\sigma(\eta - \eta')/2]} \\ &= \frac{-\sigma^2 (\hbar c / \pi)}{4\xi^2 \sinh^2[\sigma(\eta - \eta')/2]} \\ &= \frac{-\sigma^2 (\hbar c / \pi)}{4 \xi^2 \sinh^2[c\sigma(\tau - \tau')/(2\xi)]}. \end{aligned} \quad (13)$$

We notice that for very short times  $\eta - \eta' \ll 1/\sigma$ , the correlation function in Eq. (13) still goes over to the dependence  $\xi^{-2}(\eta - \eta')^{-2} = [c(\tau - \tau')]^{-2}$  appropriate for zero-point radiation in the local momentarily comoving reference frame. However, now this short-time for  $\eta - \eta'$  must be small compared to both the transition time  $\eta_{tr} = 1$  associated with the use of noninertial Rindler coordinates and also small compared to  $1/\sigma$  associated with the thermal contribution to the correlation function.

In a general coordinate frame, the local absolute temperature  $T$  in the region of thermal equilibrium follows the Tolman-Ehrenfest relation[7]  $T(g_{00})^{1/2} = \text{const.}$  In a Minkowski frame where  $g_{M00} = 1$  is a constant, the temperature  $T$  is uniform throughout the region.

However, in a noninertial frame, the local temperature  $T$  will vary with the spatial coordinate value. For a Rindler frame,  $(g_{00})^{1/2} = \xi$ , so that  $T\xi$  equals a global constant throughout the region of thermal equilibrium. For the situation of our analysis where zero-point radiation in the Rindler frame is transformed by a time-dilating conformal transformation into thermal radiation at positive absolute temperature, the scale factor  $\sigma$  provides a global empirical temperature so that the Tolman-Ehrenfest relation takes the form  $T\xi = h(\sigma)$  where  $h(\sigma)$  is some function of  $\sigma$ .

In order to derive the spectrum of thermal radiation in an inertial Minkowski frame, we must bring the spatial coordinate point (where the thermal correlation function is evaluated) out to the asymptotic region of the noninertial frame where the proper acceleration becomes ever smaller. For a Rindler frame, this means taking the coordinate  $\xi$  to spatial infinity while holding the correlation proper-time difference  $\tau - \tau'$  fixed. For zero-point radiation in Eq. (11), the argument of the hyperbolic sine function involves  $(\tau - \tau')/\xi$  and so becomes ever smaller as  $\xi \rightarrow \infty$ ; this brings the correlation function back to the *zero-point* radiation correlation function expression involving  $(\tau - \tau')^{-2}$ , exactly as in the correlation function (1) appropriate for Minkowski spacetime. In order to derive the *thermal* spectrum at positive temperature in a Minkowski frame, we must hold the local temperature fixed at a constant value in Eq. (13) by increasing  $\sigma$  as we increase the coordinate  $\xi$  so that  $\sigma/\xi$  is held constant as the coordinate point is taken to the asymptotic region. Once we are in the asymptotic region where the proper acceleration has become negligible, the only contribution to the correlation function is due to the thermal radiation spectrum. Therefore we simply carry out the Fourier time transform to obtain the thermal spectrum associated with the correlation function to find

$$\mathcal{E}_\sigma(\omega) = \frac{1}{2} \hbar \omega \coth \left( \frac{\pi \xi \omega}{c \sigma} \right). \quad (14)$$

We can check the result in Eq. (14) by going through the more familiar calculation of the correlation function of the field when starting from a known energy per normal mode.[2] Thus the relativistic radiation field is given as

$$\phi(ct, x, y, z) = \sum_{n_x, n_y, n_z} \left( \frac{8\pi k^2 \mathcal{E}}{L_x L_y L_z} \right)^{1/2} \cos[\mathbf{k} \cdot \mathbf{r} - ckt - \theta(\mathbf{k})] \quad (15)$$

where  $\mathcal{E}$  is the energy per normal mode,  $\theta(\mathbf{k})$  is the random phase associated with wavevector  $\mathbf{k}$ , and we are using box normalization where  $\mathbf{k} = \hat{x}(n_x 2\pi/L_x) + \hat{y}(n_y 2\pi/L_y) + \hat{z}(n_z 2\pi/L_z)$  so



that  $n_x, n_y, n_z$  run over all positive and negative integers. When calculating the correlation function  $\langle \phi(ct, \mathbf{r}) \phi(ct', \mathbf{r}) \rangle$ , we average over the random phases, and then assume that the box is so large that the sum over the values of  $\mathbf{k}$  may be replaced by an integral  $d^3k$ . Next we integrate over the angular directions associated with  $\mathbf{k}$ . [2] If we assume that the energy per normal mode is as given in Eq. (14), then the needed integral involves  $\int_0^\infty dx x \coth(x/2) \cos(bx) = -\pi^2 \text{sech}^2(\pi b)$  which can be obtained by taking a derivative with respect to  $b$  of  $\int_0^\infty dx \coth(x/2) \sin(bx) = \pi \coth(\pi b)$ . This last integral may be obtained from the singular Fourier sine transform  $\int_0^\infty dx \sin(bx) = 1/b$  and the standard integral  $\int_0^\infty dx \sin(bx)/[e^x - 1] = (\pi/2) \coth(\pi b)$ . [8]

The constant  $c\sigma/\xi$  appearing in Eq. (14) can be related to the temperature  $T$  by taking the small-frequency limit  $\omega \rightarrow 0$  and using the expansion  $\coth x = 1/x + x/3 - x^3/45 + \dots$  to obtain  $(1/2)\hbar c\sigma/(\pi\xi) = k_B T$ . Therefore the thermal energy spectrum in a Minkowski frame is the familiar Planck spectrum including zero-point radiation

$$\mathcal{E}(\omega, T) = \frac{1}{2}\hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) = \frac{\hbar\omega}{\exp[\hbar\omega/(k_B T)] - 1} + \frac{1}{2}\hbar\omega. \quad (16)$$

The Planck spectrum follows directly from the structure of relativistic spacetime.

It is worth emphasizing that the physicists who struggled with the problem of the black-body radiation spectrum at the turn of the 20th century were unfamiliar with two crucial ingredients necessary for a classical derivation of the Planck spectrum. These missing ingredients include the idea of classical zero-point radiation and the importance of relativity. [1] These aspects are still not understood in the textbooks of the present era where the possibility of classical zero-point radiation is never mentioned, and where classical radiation equilibrium is calculated erroneously from incompatible mixtures of nonrelativistic and relativistic ideas. [9]

The work described in this article has mathematical connections to the Unruh effect. [10] However, the analysis here is purely within classical physics and represents totally different physics from the claims of quantum field theory.

Most physicists are unaware of one of the foundational relations in Nature when they do not realize that within classical physics, the Planck blackbody spectrum reflects the

structure of relativistic spacetime.

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